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ORIGINAL RESEARCH

On mixed g-monotone and w-compatible mappings in ordered cone b-metric spaces

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Abstract In this paper, we proved common coupled fixed point results for mixed g-monotone and w-compatible mappings in ordered cone b-metric spaces. Our results extend and generalize several well-known comparable results in literature.

Keywords Cone b-metric spaces · Coupled coincidence points · Common coupled fixed points · Mixed g-monotone · w-compatible mappings · Partially ordered set

Introduction

Recently, there have been so many exciting developments in the field of existence of fixed point in partially ordered metric spaces. This trend was started by Ran and Reurings in [1], where they extended the Banach contraction principle in partially ordered sets with some applications to matrix equations. The obtained result in [1] was further extended and refined by many authors (see, for example [2, 3] and the references cited in therein).

The coupled fixed point theorem is an interesting and decisive concept in fixed point theory. In 2006, Bhaskar and Lakshmikantham [4] introduced the notions of the coupled fixed point and mixed monotone property of a given mapping $F: X \times X \longrightarrow X$, where X is a nonempty ordered set. In addition, they proved some coupled fixed point theorems for mappings which satisfy the mixed monotone property and considered some applications in the existence and uniqueness of a solution for a periodic boundary value problem. They also established the classical coupled fixed point theorems and obtained some of their applications.

Inspired by above notions, Lakshmikantham and Ćirić in [2] introduced the concepts of the coupled coincidence point, the common coupled fixed point and g-mixed monotone property for mappings $F: X \times X \longrightarrow X$ and $g: X \longrightarrow X$. In subsequent papers several authors proved various coupled and common coupled fixed point theorems (e.g., [5–14]).

On the other hand, Huang and Zhang [15] reintroduced the notion of cone metric spaces and established fixed point theorems for mappings in such spaces. Next in 2011, Hussain and Shah [16] (see also [17]) generalized the notion of cone metric spaces and introduced cone b-metric spaces. Afterwards, many authors obtained many fixed point, common fixed point and common coupled fixed point theorems in cone b-metric spaces. For some works in cone b-metric spaces, we may refer the reader to [18–29].

In 2012, Nashine et al. [3] established coupled coincidence point results for mixed g-monotone mappings under

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general contractive conditions in partially ordered cone metric spaces over solid cones. For more results on ordered metric spaces, cone and ordered cone metric spaces, the reader can be traced back to [30–39].

Theorem 1.1 [3] *Let $(X, d, \sqsubseteq)$ be an ordered cone metric space over a solid cone P. Let $F : X^2 \longrightarrow X$ and $g : X \longrightarrow X$ be mappings such that F has the mixed g-monotone property on X and there exist two elements $x_0, y_0 \in X$ with $gx_0 \sqsubseteq F(x_0, y_0)$ and $gy_0 \sqsupseteq F(y_0, x_0)$. Suppose further that F, g satisfy*

$$\begin{aligned} d\big(F(x,y),F(u,v)\big) \preceq a_1 d(gx,gu) &+ a_2 d\big(F(x,y),gx\big)\\ &+ a_3 d(gy,gv)\\ &+ a_5 d\big(F(x,y),gu\big)\\ &+ a_6 d\big(F(u,v),gx\big), \end{aligned}$$

for all $(x,y),(u,v) \in X \times X$ with ($gu \sqsubseteq gx$ and $gv \sqsupseteq gy$) or ($gx \sqsubseteq gu$ and $gy \sqsupseteq gv$), where $a_i \geq 0$, for i =1,2,...,6 and $\sum_{i=1}^{6} a_i < 1$. Further suppose

1. *$F(X \times X) \subseteq g(X)$;*
2. *$g(X)$ is a complete subspace of X.*

Also, suppose that X has the following properties :

(i) *if a non-decreasing sequence $\{x_n\}$* in X is such that $x_n \longrightarrow x$,then $x_n \sqsubseteq x$ for all $n \in N$,

(ii) if a non-increasing sequence $\{y_n\}$ in X is such that $y_n \longrightarrow y$,then $y_n \sqsupseteq y$ for all $n \in N$.

Then there exist x and y such that $F(x,y) = gx$ and $F(y,x) = gy$, that is, F and g have a coupled coincidence point $(x,y) \in X \times X$.

Theorem 1.2 [3] *In addition to the hypotheses of Theorem 1.1, suppose that for every $(x,y),(y^*,x^*) \in X \times X$ there exists $(u,v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable both to $(F(x, y), F(y, x))$ and $(F(x^*,y^*), F(y^*,x^*))$. Then F and g have a unique coupled common fixed point, that is, there exists a unique $(u,v) \in X \times X$ such that*

$$u = gu = F(u,v) \quad \text{and} \quad v = gv = F(v,u),$$

provided F and g are w^*-compatible.

In this paper, an essay is made to establish common coupled fixed point results for mixed g-monotone and w-compatible mappings satisfying more general contractive conditions in ordered cone b-metric spaces over a cone that is only solid.

Preliminaries

Let E be a real Banach space and θ denote to the zero element in E. A cone P is a subset of E such that:

1. P is nonempty closed set and $P \neq \{\theta\}$,
2. If a, b are nonnegative real numbers and $x, y \in P$ then $ax + by \in P$,
3. $x \in P$ and $-x \in P$ imply $x = \theta$.

For any cone $P \subset E$, the partial ordering $\preceq$ with respect to P is defined by $x \preceq y$ if and only if $y - x \in P$. The notation of $\prec$ stand for $x \preceq y$ but $x \neq y$. Also, we used $x \ll y$ to indicate that $y - x \in \mathrm{int}P$, where $\mathrm{int}P$ denotes the interior of P. A cone P is called normal if there exists a number K such that

$$\theta \preceq x \preceq y \Longrightarrow \|x\| \leq K\|y\|,$$

for all $x, y \in E$. The least positive number K satisfying the above condition is called the normal constant of P. Throughout this paper, we do not impose the normality condition for the cones, but the only assumption is that the cone P is solid, that is $\mathrm{int}P \neq \emptyset$.

Definition 2.1 ([16]) Let X be a nonempty set and E be a real Banach space equipped with the partial ordering $\preceq$ with respect to the cone P. A vector-valued function $d : X \times X -\!\!\rightarrow E$ is said to be a cone b-metric function on X with the constant $s \geq 1$ if the following conditions are satisfied:

1. $\theta \preceq d(x,y)$ for all $x, y \in X$ and $d(x,y) = \theta$ if and only if $x = y$,
2. $d(x,y) = d(y,x)$ for all $x, y \in X$,
3. $d(x,y) \preceq s(d(x,y) + \mathrm{d}(y,z))$ for all $x, y, z \in X$.

Then pairs (X, d) is called a cone b-metric space (or a cone metric type space), we will use the first mentioned term.

Definition 2.2 ([16]) Let (X, d) be a cone b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$.

1. For all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $d(x_n, x) \ll c$ for all $n > N$, then x_n is said to be convergent and x is the limit of $\{x_n\}$. We denote this by $x_n \longrightarrow x$.
2. For all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $d(x_n, x_m) \ll c$ for all $n, m > N$, then $\{x_n\}$ is called a Cauchy sequence in X.
3. A cone b-metric space (X, d) is called complete if every Cauchy sequence in X is convergent.

The following lemma is helpful to prove our results.

Lemma 2.3 ([40])

1. If *E be a real Banach space with a cone P and $a \preceq \lambda a$ where $a \in P$ and* $0 \leq \lambda < 1$, *then $a = \theta$.*
2. If $c \in \mathrm{int}P$, *$\theta \preceq a_n$ and $a_n \longrightarrow \theta$, then there exists a positive integer N such that $a_n \ll c$ for all $n \geq N$.*
3. If *$a \preceq b$ and $b \ll c$, then $a \ll c$.*
4. If *$\theta \preceq u \ll c$ for each $\theta \ll c$, then $u = \theta$.*

Recall the following definitions.

Definition 2.4 ([4]) Let $(X, \sqsubseteq)$ be a partially ordered set and let $F : X^2 \longrightarrow X$ be a mapping. The mapping F is said to have mixed monotone property if F is monotone non-decreasing in its first argument and monotone non-increasing in its second argument, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \quad x_1 \sqsubseteq x_2 \Longrightarrow F(x_1, y) \sqsubseteq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad y_1 \sqsubseteq y_2 \Longrightarrow F(x, y_1) \sqsupseteq F(x, y_2).$$

Definition 2.5 ([2]) Let $(X, \sqsubseteq)$ be a partially ordered set and let $F : X^2 \longrightarrow X$ and $g : X \longrightarrow X$ be two mappings. The mapping F is said to have mixed g-monotone property if F is monotone g-non-decreasing in its first argument and monotone g-non-increasing in its second argument, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \quad gx_1 \sqsubseteq gx_2 \Longrightarrow F(x_1, y) \sqsubseteq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad gy_1 \sqsubseteq gy_2 \Longrightarrow F(x, y_1) \sqsupseteq F(x, y_2).$$

Definition 2.6 ([4]) An element $(x, y) \in X^2$ is said to be a coupled fixed point of the mapping $F : X^2 \longrightarrow X$ if $F(x, y) = x$ and $F(y, x) = y$.

Definition 2.7 ([2]) An element $(x, y) \in X^2$ is called

1. a coupled coincidence point of mappings $F : X^2 \longrightarrow X$ and $g : X \longrightarrow X$ if $gx = F(x, y)$ and $gy = F(y, x)$, and (gx, gy) is called coupled point of coincidence.
2. a common coupled fixed point of mappings $F : X^2 \longrightarrow X$ and $g : X \longrightarrow X$ if $x = gx = F(x, y)$ and $y = gy = F(y, x)$.

Definition 2.8 ([5]) The mappings $F : X^2 \longrightarrow X$ and $g : X \longrightarrow X$ are called:

1. w-compatible if $g(F(x, y)) = F(gx, gy)$ whenever $gx = F(x, y)$ and $gy = F(y, x)$.
2. w^*-compatible if $g(F(x, x)) = F(gx, gx)$ whenever $gx = F(x, x)$.

Coupled coincidence point and common coupled fixed point results

In this section, we prove some coupled coincidence point and common coupled fixed point results in ordered cone b-metric spaces.

Theorem 3.1 *Let $(X, \sqsubseteq)$ (be a partially ordered set and X, d be a cone b-metric space with the coefficient) $s \geq 1$ relative to a solid cone P. Let $F : X^2 \longrightarrow X$ and $g : X \longrightarrow X$ be two mappings such that F has the mixed g-monotone property on X and suppose that there exist nonnegative constants $a_i \in [0, 1), i = 1, 2, \ldots, 10$ with $(s+1)(a_1 + a_2 + a_3 + a_4) + s(s+1) \quad (a_5 + a_6 + a_7 + a_8) + 2s(a_9 + a_{10}) < 2$ and $\sum_{i=1}^{10} a_i < 1$ such that the following contractive condition holds*

$$\begin{aligned} d(F(x,y),F(u,v)) \preceq {} & [a_1 d(gx, F(x,y)) + a_2 d(gy, F(y,x))] \\ & + [a_3 d(gu, F(u,v)) + a_4 d(gv, F(v,u))] \\ & + [a_5 d(gx, F(u,v)) + a_6 d(gy, F(v,u))] \\ & + [a_7 d(gu, F(x,y)) + a_8 d(gv, F(y,x))] \\ & + [a_9 d(gx, gu) + a_{10} d(gy, gv)], \end{aligned}$$

for all $(x, y), (u, v) \in X^2$ with ($gu \sqsubseteq gx$ and $gv \sqsupseteq gy$) or ($gx \sqsubseteq gu$ and $gy \sqsupseteq gv$). Assume that F and g satisfy the following conditions:

1. $F(X^2) \subseteq g(X)$,
2. *$g(X)$ is a complete subspace of X.*

Also, suppose that X has the following properties:

(i) if a non-decreasing sequence $\{x_n\}$ in X is such that $x_n \longrightarrow x$, then $x_n \sqsubseteq x$ for all $n \in \mathbb{N}$,
(ii) if a non-increasing sequence $\{y_n\}$ in X is such that $y_n \longrightarrow y$, then $y_n \sqsupseteq y$ for all $n \in \mathbb{N}$.

If there exist $x_0, y_0 \in X$ such that $gx_0 \sqsubseteq F(x_0, y_0)$ and $F(y_0, x_0) \sqsubseteq gy_0$, then F and g have a coupled coincidence point $(x^*, y^*) \in X^2$.

Proof Let $x_0, y_0 \in X$ such that $gx_0 \sqsubseteq F(x_0, y_0)$ and $F(y_0, x_0) \sqsubseteq gy_0$. Since $F(X^2) \subseteq g(X)$ we can Choose $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0), gy_1 = F(y_0, x_0)$. Again Since $F(X^2) \subseteq g(X)$ we can Choose $x_2, y_2 \in X$ such that $gx_2 = F(x_1, y_1), gy_2 = F(y_1, x_1)$. Since F has the mixed g-monotone property, we have $gx_0 \sqsubseteq gx_1 \sqsubseteq gx_2$ and $gy_2 \sqsubseteq gy_1 \sqsubseteq gy_0$. Continuing this process, we can construct two sequences $\{x_n\}, \{y_n\}$ in X such that

$$gx_n = F(x_{n-1}, y_{n-1}) \sqsubseteq gx_{n+1} = F(x_n, y_n)$$

and

$$gy_{n+1} = F(y_n, x_n) \sqsubseteq gy_n = F(y_{n-1}, x_{n-1}).$$

Then we have:

$$\begin{aligned} & d(gx_n, gx_{n+1}) \\ & = d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ & \preceq [a_1 d(gx_{n-1}, F(x_{n-1}, y_{n-1}) + a_2 d(gy_{n-1}, F(y_{n-1}, x_{n-1}))] \\ & \quad + [a_3 d(gx_n, F(x_n, y_n)) + a_4 d(gy_n, F(y_n, x_n))] \\ & \quad + [a_5 d(gx_{n-1}, F(x_n, y_n)) + a_6 d(gy_{n-1}, F(y_n, x_n))] \\ & \quad + [a_7 d(gx_n, F(x_{n-1}, y_{n-1})) + a_8 d(gy_n, F(y_{n-1}, x_{n-1}))] \\ & \quad + [a_9 d(gx_{n-1}, gx_n) + a_{10} d(gy_{n-1}, gy_n)]. \end{aligned}$$

So that,

$$\begin{aligned}d(gx_n, gx_{n+1}) &= d\big(F(x_{n-1}, y_{n-1}), F(x_n, y_n)\big)\\ &\preceq \big[a_1 d(gx_{n-1}, gx_n) + a_2 d(gy_{n-1}, gy_n)\big]\\ &\quad + \big[a_3 d(gx_n, gx_{n+1}) + a_4 d(gy_n, gy_{n+1})\big]\\ &\quad + \big[a_5 d(gx_{n-1}, gx_{n+1}) + a_6 d(gy_{n-1}, gy_{n+1})\big]\\ &\quad + \big[a_7 d(gx_n, gx_n) + a_8 d(gy_n, gy_n)\big]\\ &\quad + \big[a_9 d(gx_{n-1}, gx_n) + a_{10} d(gy_{n-1}, gy_n)\big]\\ &\preceq \big[a_1 d(gx_{n-1}, gx_n) + a_2 d(gy_{n-1}, gy_n)\big]\\ &\quad + \big[a_3 d(gx_n, gx_{n+1}) + a_4 d(gy_n, gy_{n+1})\big]\\ &\quad + \big[sa_5(d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1}))\\ &\quad + sa_6(d(gy_{n-1}, gy_n) + d(gy_n, gy_{n+1}))\big]\\ &\quad + \big[a_9 d(gx_{n-1}, gx_n) + a_{10} d(gy_{n-1}, gy_n)\big].\end{aligned}$$

Hence

$$\begin{aligned}d(gx_n, gx_{n+1}) \preceq{}& \big[(a_1 + a_5 s + a_9) d(gx_{n-1}, gx_n)\\ &+ (a_2 + a_6 s + a_{10}) d(gy_{n-1}, gy_n)\big]\\ &+ \big[(a_3 + a_5 s) d(gx_n, gx_{n+1})\\ &+ (a_4 + a_6 s) d(gy_n, gy_{n+1})\big].\end{aligned} \tag{3.1}$$

Similarly, we can prove that

$$\begin{aligned}d(gy_n, gy_{n+1}) \preceq{}& \big[(a_1 + sa_5 + a_9) d(gy_{n-1}, gy_n)\\ &+ (a_2 + sa_6 + a_{10}) d(gx_{n-1}, gx_n)\big]\\ &+ \big[(a_3 + sa_5) d(gy_n, gy_{n+1})\\ &+ (a_4 + sa_6) d(gx_n, gx_{n+1})\big].\end{aligned} \tag{3.2}$$

Put,

$$d_n = d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}).$$

Adding inequalities (3.1) and (3.2), one can assert that

$$\begin{aligned}d_n \preceq{}& (a_1 + a_2 + sa_5 + sa_6 + a_9 + a_{10}) d_{n-1}\\ &+ (a_3 + a_4 + sa_5 + sa_6) d_n.\end{aligned} \tag{3.3}$$

On the other hand, we have

$$\begin{aligned}&d(gx_{n+1}, gx_n) = d\big(F(x_n, y_n), F(x_{n-1}, y_{n-1})\big)\\ &\preceq \big[a_1 d(gx_n, F(x_n, y_n)) + a_2 d(gy_n, F(y_n, x_n))\big]\\ &\quad + \big[a_3 d(gx_{n-1}, F(x_{n-1}, y_{n-1}) + a_4 d(gy_{n-1}, F(y_{n-1}, x_{n-1}))\big]\\ &\quad + \big[a_5 d(gx_n, F(x_{n-1}, y_{n-1})) + a_6 d(gy_n, F(y_{n-1}, x_{n-1}))\big]\\ &\quad + \big[a_7 d(gx_{n-1}, F(x_n, y_n)) + a_8 d(gy_{n-1}, F(y_n, x_n))\big]\\ &\quad + \big[a_9 d(gx_n, gx_{n-1}) + a_{10} d(gy_n, gy_{n-1})\big].\end{aligned}$$

So that,

$$\begin{aligned}d(gx_{n+1}, gx_n) &= d\big(F(x_n, y_n), F(x_{n-1}, y_{n-1})\big)\\ &\preceq \big[a_1 d(gx_n, gx_{n+1}) + a_2 d(gy_n, gy_{n+1})\big]\\ &\quad + \big[a_3 d(gx_{n-1}, gx_n) + a_4 d(gy_{n-1}, gy_n)\big]\\ &\quad + \big[a_5 d(gx_n, gx_n) + a_6 d(gy_n, gy_n)\big]\\ &\quad + \big[a_7 d(gx_{n-1}, gx_{n+1}) + a_8 d(gy_{n-1}, gy_{n+1})\big]\\ &\quad + \big[a_9 d(gx_n, gx_{n-1}) + a_{10} d(gy_n, gy_{n-1})\big]\\ &\preceq \big[a_1 d(gx_n, gx_{n+1}) + a_2 d(gy_n, gy_{n+1})\big]\\ &\quad + \big[a_3 d(gx_{n-1}, gx_n) + a_4 d(gy_{n-1}, gy_n)\big]\\ &\quad + \big[sa_7(d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1}))\\ &\quad + sa_8(d(gy_{n-1}, gy_n) + d(gy_n, gy_{n+1}))\big]\\ &\quad + \big[a_9 d(gx_{n-1}, gx_n) + a_{10} d(gy_{n-1}, gy_n)\big].\end{aligned}$$

Hence

$$\begin{aligned}d(gx_{n+1}, gx_n) \preceq{}& \big[(a_3 + sa_7 + a_9) d(gx_{n-1}, gx_n)\\ &+ (a_4 + sa_8 + a_{10}) d(gy_{n-1}, gy_n)\big]\\ &+ \big[(a_1 + sa_7) d(gx_n, gx_{n+1})\\ &+ (a_2 + sa_8) d(gy_n, gy_{n+1})\big].\end{aligned} \tag{3.4}$$

Similarly

$$\begin{aligned}d(gy_{n+1}, gy_n) \preceq{}& \big[(a_3 + sa_7 + a_9) d(gy_{n-1}, gy_n)\\ &+ (a_4 + sa_8 + a_{10}) d(gx_{n-1}, gx_n)\big]\\ &+ \big[(a_1 + sa_7) d(gy_n, gy_{n+1})\\ &+ (a_2 + sa_8) d(gx_n, gx_{n+1})\big].\end{aligned} \tag{3.5}$$

Adding inequalities (3.4) and (3.5), one can assert that

$$\begin{aligned}d_n \preceq{}& (a_3 + a_4 + sa_7 + sa_8 + a_9 + a_{10}) d_{n-1}\\ &+ (a_1 + a_2 + sa_7 + sa_8) d_n.\end{aligned} \tag{3.6}$$

Finally, from (3.3) and (3.6), we have

$$\begin{aligned}2d_n \preceq{}& (a_1 + a_2 + a_3 + a_4 + sa_5 + sa_6 + sa_7 + sa_8\\ &+ 2(a_9 + a_{10})) d_{n-1} + (a_1 + a_2 + a_3 + a_4 + sa_5\\ &+ sa_6 + sa_7 + sa_8) d_n,\end{aligned}$$

that is

$$d_n \preceq h d_{n-1},$$

where $h = \frac{\big(a_1+a_2+a_3+a_4+sa_5+sa_6+sa_7+sa_8+2(a_9+a_{10})\big)}{2-(a_1+a_2+a_3+a_4+sa_5+sa_6+sa_7+sa_8)} < \frac{1}{s}$.

Note that, $h = \frac{\big(a_1+a_2+a_3+a_4+sa_5+sa_6+sa_7+sa_8+2(a_9+a_{10})\big)}{2-(a_1+a_2+a_3+a_4+sa_5+sa_6+sa_7+sa_8)} < \frac{1}{s}$ equivalently $(s+1)(a_1 + a_2 + a_3 + a_4) + s(s+1)\ \ (a_5 + a_6 + a_7 + a_8) + 2s(a_9 + a_{10}) < 2$.

Consequently, we have

$$\begin{aligned}d_n &\preceq h d_{n-1}\\ &\preceq h^2 d_{n-2}\\ &\preceq h^3 d_{n-3}\\ &\ \ \vdots\\ &\preceq h^n d_0.\end{aligned} \tag{3.7}$$

Let $m > n \geq 1$. It follows that

$$\begin{aligned}d(gx_n, gx_m) \preceq{}& sd(gx_n, gx_{n+1}) + s^2 d(gx_{n+1}, gx_{n+2})\\ &+ \cdots + s^{m-n} d(gx_{m-1}, gx_m),\end{aligned}$$

and

$$d(gy_n, gy_m) \preceq sd(gy_n, gy_{n+1}) + s^2 d(gy_{n+1}, gy_{n+2}) + \cdots + s^{m-n} d(gy_{m-1}, gy_m).$$

Now, (3.7) and $sh < 1$ imply that

$$\begin{aligned} d(gx_n, gx_m) + d(gy_n, gy_m) &\preceq sd_n + s^2 d_{n+1} + \cdots + s^{m-n} d_{m-1} \\ &\preceq sh^n d_0 + s^2 h^{n+1} d_0 + \cdots + s^{m-n} h^{m-1} d_0 \\ &= sh^n (1 + sh + (sh)^2 + \cdots + (sh)^{m-n-1}) d_0 \\ &\preceq \frac{sh^n}{1 - sh} d_0 \quad \to \theta \quad \text{as} \quad n \to \infty. \end{aligned} \tag{3.8}$$

According to Lemma 2.3 (2), and for any $c \in E$ with $c \gg \theta$, there exists $N_0 \in \mathbb{N}$ such that for any $n > N_0$, $\frac{h^n}{1-h} d_0 \ll c$. Furthermore, from (3.8) and for any $m > n > N_0$, Lemma 2.3 (3) shows that

$$d(gx_n, gx_m) + d(gy_n, gy_m) \ll c,$$

which implies that

$$d(gx_n, gx_m) \ll c,$$

and

$$d(gy_n, gy_m) \ll c.$$

Hence, by Definition 2.2 (2), $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in $g(X)$. Since $g(X)$ is complete, there exists x^* and $y^* \in X$ such that $gx_n \longrightarrow gx^*$ and $gy_n \longrightarrow gy^*$ as $n \longrightarrow \infty$. Since $\{gx_n\}$is nondecreasing and $\{gy_n\}$ is non-increasing, using the properties (i), (ii) of X, we have

$$gx_n \sqsubseteq gx^* \quad \textit{and} \quad gy^* \sqsubseteq gy_n.$$

Now, we can apply the contractive condition

$$\begin{aligned} d(F(x^*, y^*), gx^*) &\preceq s(d(F(x^*, y^*), gx_{n+1}) + d(gx_{n+1}, gx^*)) \\ &= s(d(F(x^*, y^*), F(x_n, y_n)) + d(gx_{n+1}, gx^*)) \\ &\preceq s\big[a_1 d(gx^*, F(x^*, y^*)) + a_2 d(gy^*, F(y^*, x^*))\big] \\ &\quad + s\big[a_3 d(gx_n, F(x_n, y_n)) + a_4 d(gy_n, F(y_n, x_n))\big] \\ &\quad + s\big[a_5 d(gx^*, F(x_n, y_n)) + a_6 d(gy^*, F(y_n, x_n))\big] \\ &\quad + s\big[a_7 d(gx_n, F(x^*, y^*)) + a_8 d(gy_n, F(y^*, x^*))\big] \\ &\quad + s\big[a_9 d(gx^*, gx_n) + a_{10} d(gy^*, gy_n)\big] + sd(gx_{n+1}, gx^*) \\ &\preceq s\big[a_1 d(F(x^*, y^*), gx^*) + a_2 d(F(y^*, x^*), gy^*)\big] \\ &\quad + s\big[sa_3 d(gx_n, gx^*) + sa_3 d(gx^*, gx_{n+1}) \\ &\quad + sa_4 d(gy_n, gy^*) + sa_4 d(gy^*, gy_{n+1})\big] \\ &\quad + s\big[a_5 d(gx^*, gx_{n+1}) + a_6 d(gy^*, gy_{n+1})\big] \\ &\quad + s\big[sa_7 d(gx_n, gx^*) + sa_7 d(gx^*, F(x^*, y^*)) \\ &\quad + sa_8 d(gy_n, gy^*) + sa_8 d(gy^*, F(y^*, x^*))\big] \\ &\quad + s\big[a_9 d(gx^*, gx_n) + a_{10} d(gy^*, gy_n)\big] + sd(gx_{n+1}, gx^*) \\ &= s\big[a_1 d(F(x^*, y^*), gx^*) + a_2 d(F(y^*, x^*), gy^*)\big] \\ &\quad + s\big[sa_3 d(gx_n, gx^*) + sa_3 d(gx_{n+1}, gx^*) \\ &\quad + sa_4 d(gy_n, gy^*) + sa_4 d(gy_{n+1}, gy^*)\big] \\ &\quad + s\big[a_5 d(gx_{n+1}, gx^*) + a_6 d(gy_{n+1}, gy^*)\big] \\ &\quad + s\big[sa_7 d(gx_n, gx^*) + sa_7 d(F(x^*, y^*), gx^*) \\ &\quad + sa_8 d(gy_n, gy^*) + sa_8 d(F(y^*, x^*), gy^*)\big] \\ &\quad + s\big[a_9 d(gx_n, gx^*) + a_{10} d(gy_n, gy^*)\big] + sd(gx_{n+1}, gx^*). \end{aligned}$$

Hence,

$$\begin{aligned} d(F(x^*, y^*), gx^*) \preceq\ & (sa_1 + s^2 a_7) d(F(x^*, y^*), gx^*) \\ & + (a_2 + s^2 a_8) d(F(y^*, x^*), gy^*) \\ & + (s^2 a_3 + s^2 a_7 + sa_9) d(gx_n, gx^*) \\ & + (s^2 a_3 + sa_5 + s) d(gx_{n+1}, gx^*) \\ & + (s^2 a_4 + s^2 a_8 + sa_{10}) d(gy_n, gy^*) \\ & + (s^2 a_4 + sa_6) d(gy_{n+1}, gy^*). \end{aligned}$$

Similarly

$$\begin{aligned} d(F(y^*, x^*), gy^*) \preceq\ & (sa_1 + s^2 a_7) d(F(y^*, x^*), gy^*) \\ & + (a_2 + s^2 a_8) d(F(x^*, y^*), gx^*) \\ & + (s^2 a_3 + s^2 a_7 + sa_9) d(gy_n, gy^*) \\ & + (s^2 a_3 + sa_5 + s) d(gy_{n+1}, gy^*) \\ & + (s^2 a_4 + s^2 a_8 + sa_{10}) d(gx_n, gx^*) \\ & + (s^2 a_4 + sa_6) d(gx_{n+1}, gx^*). \end{aligned}$$

Put

$$\tau = d(F(x^*, y^*), gx^*) + d(F(y^*, x^*), gy^*).$$

Adding above inequalities, we get

$$\begin{aligned} \tau \preceq\ & (sa_1 + s^2 a_7 + a_2 + s^2 a_8) \tau \\ & + (s^2 a_3 + s^2 a_7 + sa_9 + s^2 a_4 + s^2 a_8 + sa_{10}) d(gx_n, gx^*) \\ & + (s^2 a_3 + s^2 a_7 + sa_9 + s^2 a_4 + s^2 a_8 + sa_{10}) d(gy_n, gy^*) \\ & + (s^2 a_3 + sa_5 + s + s^2 a_4 + sa_6) d(gx_{n+1}, gx^*) \\ & + (s^2 a_3 + sa_5 + s + s^2 a_4 + sa_6) d(gy_{n+1}, gy^*). \end{aligned}$$

Then,

$$\begin{aligned} \tau \preceq\ & \frac{A_2}{1 - A_1} d(gx_n, gx^*) + \frac{A_2}{1 - A_1} d(gy_n, gy^*) \\ & + \frac{A_3}{1 - A_1} d(gx_{n+1}, gx^*) + \frac{A_3}{1 - A_1} d(gy_{n+1}, gy^*), \end{aligned}$$

where $A_1 = sa_1 + s^2 a_7 + a_2 + s^2 a_8$, $A_2 = s^2 a_3 + s^2 a_7 + sa_9 + s^2 a_4 + s^2 a_8 + sa_{10}$ and $A_3 = s^2 a_3 + sa_5 + s + s^2 a_4 + sa_6$. Since $gx_n \longrightarrow gx^*$ and $gy_n \longrightarrow gy^*$ as $n \longrightarrow \infty$, then by Definition 2.2 (1) and for $c \gg \theta$ there exists $N_0 \in \mathbb{N}$ such that for all $n > N_0$, $d(gx_n, gx^*) \ll c\frac{1-A_1}{4A_2}$, $d(gy_n, gy^*) \ll c\frac{1-A_1}{4A_2}$, $d(gx_{n+1}, gx^*) \ll c\frac{1-A_1}{4A_3}$ and $d(gy_{n+1}, gy^*) \ll c\frac{1-A_1}{4A_3}$. Hence,

$$\begin{aligned}\tau &\preceq \frac{A_2}{1-A_1}d(gx_n, gx^*) + \frac{A_2}{1-A_1}d(gy_n, gy^*)\\ &\quad + \frac{A_3}{1-A_1}d(gx_{n+1}, gx^*) + \frac{A_3}{1-A_1}d(gy_{n+1}, gy^*)\\ &\ll c\frac{1-A_1}{4A_2}\frac{A_2}{1-A_1} + c\frac{1-A_1}{4A_2}\frac{A_2}{1-A_1} + c\frac{1-A_1}{4A_3}\frac{A_3}{1-A_1}\\ &\quad + c\frac{1-A_1}{4A_3}\frac{A_3}{1-A_1} = c.\end{aligned}$$

Now, according to Lemma 2.3 (4) it follows that $\tau = \theta$, that is, $d(F(x^*, y^*), gx^*) + d(F(y^*, x^*), gy^*) = \theta$, which implies that $d(F(x^*, y^*), gx^*) = \theta$ and $d(F(y^*, x^*), gy^*) = \theta$. Hence, $gx^* = F(x^*, y^*)$ and $gy^* = F(y^*, x^*)$. Therefore (x^*, y^*) is a coupled coincidence point of F and g. □

From Theorem 3.1, we have the following corollaries.

Corollary 3.2 Let $(X, \sqsubseteq)$ *(be a partially ordered set and* X, d *be a cone b-metric space with the coefficient)* $s \geq 1$ *relative to a solid cone* P. *Let* $F : X^2 \longrightarrow X$ *and* $g : X \longrightarrow X$ *be two mappings and suppose that there exist nonnegative constants* $k, l \in (0, 1]$ *with* $k + l < \frac{1}{s}$ *such that the following contractive condition holds for all* $x, y, u, v \in X$:

$$d\big(F(x, y), F(u, v)\big) \preceq kd(gx, gu) + ld(gy, gv),$$

for all $(x, y), (u, v) \in X^2$ *with* $(gu \sqsubseteq gx$ *and* $gv \sqsupseteq gy)$ *or* $(gx \sqsubseteq gu$ *and* $gy \sqsupseteq gv)$. *Assume that* F *and* g *satisfy the following conditions*:

1. $F(X^2) \subseteq g(X)$,
2. $g(X)$ *is a complete subspace of* X.

Also, suppose that X has the following properties:

(i) if a non-decreasing sequence $\{x_n\}$ in X is such that $x_n \longrightarrow x$, then $x_n \sqsubseteq x$ for all $n \in \mathbb{N}$,
(ii) if a non-increasing sequence $\{y_n\}$ in X is such that $y_n \longrightarrow y$, then $y_n \sqsupseteq y$ for all $n \in \mathbb{N}$.

If there exist $x_0, y_0 \in X$ such that $gx_0 \sqsubseteq F(x_0, y_0)$ and $F(y_0, x_0) \sqsubseteq gy_0$, then F and g have a coupled coincidence point $(x^*, y^*) \in X^2$.

Corollary 3.3 *Let* $(X, \sqsubseteq)$ *be a partially ordered set and* (X, d) *be a cone b-metric space with the coefficient* $s \geq 1$ *relative to a solid cone* P. *Let* $F : X^2 \longrightarrow X$ *and* $g : X \longrightarrow X$ *be two mappings and suppose that there exist nonnegative constants* $k, l \in (0, 1]$ *with* $k + l < \frac{2}{s+1}$ *such that the following contractive condition holds for all* $x, y, u, v \in X$:

$$d\big(F(x, y), F(u, v)\big) \preceq kd(gx, F(x, y)) + ld(gu, F(u, v)),$$

for all $(x, y), (u, v) \in X^2$ *with* $(gu \sqsubseteq gx$ *and* $gv \sqsupseteq gy)$ *or* $(gx \sqsubseteq gu$ *and* $gy \sqsupseteq gv)$. *Assume that* F *and* g *satisfy the following conditions*:

1. $F(X^2) \subseteq g(X)$,
2. $g(X)$ *is a complete subspace of* X.

Also, suppose that X has the following properties:

(i) if a non-decreasing sequence $\{x_n\}$ in X is such that $x_n \longrightarrow x$, then $x_n \sqsubseteq x$ for all $n \in \mathbb{N}$,
(ii) if a non-increasing sequence $\{y_n\}$ in X is such that $y_n \longrightarrow y$, then $y_n \sqsupseteq y$ for all $n \in \mathbb{N}$.

If there exist $x_0, y_0 \in X$ such that $gx_0 \sqsubseteq F(x_0, y_0)$ and $F(y_0, x_0) \sqsubseteq gy_0$, then F and g have a coupled coincidence point $(x^*, y^*) \in X^2$.

Corollary 3.4 *Let* (X, d) *be a cone b-metric space with the coefficient* $s \geq 1$ *relative to a solid cone* P. *Let* $F : X^2 \longrightarrow X$ *and* $g : X \longrightarrow X$ *be two mappings and suppose that there exist nonnegative constants* $k, l \in (0, 1]$ *with* $k + l < \frac{2}{s(s+1)}$ *such that the following contractive condition holds for all* $x, y, u, v \in X$:

$$d\big(F(x, y), F(u, v)\big) \preceq kd\big(gx, F(u, v)\big) + ld\big(gu, F(x, y)\big),$$

for all $(x, y), (u, v) \in X^2$ *with* $(gu \sqsubseteq gx$ *and* $gv \sqsupseteq gy)$ *or* $(gx \sqsubseteq gu$ *and* $gy \sqsupseteq gv)$. *Assume that* F *and* g *satisfy the following conditions*:

1. $F(X^2) \subseteq g(X)$,
2. $g(X)$ *is a complete subspace of* X.

Also, suppose that X has the following properties:

(i) if a non-decreasing sequence $\{x_n\}$ in X is such that $x_n \longrightarrow x$, then $x_n \sqsubseteq x$ for all $n \in \mathbb{N}$,
(ii) if a non-increasing sequence $\{y_n\}$ in X is such that $y_n \longrightarrow y$, then $y_n \sqsupseteq y$ for all $n \in \mathbb{N}$.

If there exist $x_0, y_0 \in X$ such that $gx_0 \sqsubseteq F(x_0, y_0)$ and $F(y_0, x_0) \sqsubseteq gy_0$, then F and g have a coupled coincidence point $(x^*, y^*) \in X^2$.

Now we prove the existence and uniqueness of a common coupled fixed point. Note that, if $(X, \sqsubseteq)$ is a partially ordered set, then we endow the product space $X \times X$ with the following partial order: for $(x, y), (u, v) \in X \times X$, $(u, v) \sqsubseteq (x, y) \Longleftrightarrow x \sqsupseteq u, y \sqsubseteq v$.

Theorem 3.5 *In addition to the hypotheses of Theorem 3.1, suppose that for every* $(x, y), (x^*, y^*) \in X \times X$ *there exists* $(u, v) \in X \times X$ *such that* $(F(u, v), F(v, u))$ *is comparable both to* $(F(x, y), F(y, x))$ *and* $(F(x^*, y^*), F(y^*, x^*))$. *Assume that,* $s(a_1 + a_2 + a_3 + a_4) + (a_5 + a_6 + a_7 + a_8 + a_9 + a_{10}) < 1$. *If* F *and* g *are w-compatible, then* F *and* g *have a unique common coupled fixed point. Moreover, a* *common coupled fixed point of F and g is of the form* (u, u) *for some* $u \in X$.

Proof From Theorem 3.1, F and g have a coupled coincidence point. Suppose (x, y) and (x^*, y^*) are coupled coincidence points of F and g, that is $gx = F(x,y), gy = F(y,x)$ and $gx^* = F(x^*,y^*), gy^* = F(y^*,x^*)$. First, we will show that

$$gx = gx^* \quad and \quad gy = gy^*.$$

By assumption, there exists $(u,v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable both to $(F(x, y), F(y, x))$ and $(F(x^*,y^*), F(y^*,x^*))$. Put $u_0 = u, v_0 = v$ and choose $u_1, v_1 \in X$ such that $gu_1 = F(u_0, v_0)$ and $gv_1 = F(v_0, u_0)$. Continuing this process, we can construct two sequences $\{gu_n\}$ and $\{gv_n\}$ such that

$$gu_{n+1} = F(u_n, v_n) \quad and \quad gv_{n+1} = F(v_n, u_n).$$

Also, set $x_0 = x, y = y_0, x_0^* = x^*, y_0^* = y^*$. Define the sequences $\{gx_n\}, \{gy_n\}$ and $\{gx_n^*\}, \{gy_n^*\}$. Since (x, y) and (x^*, y^*) are coupled coincidence points of F and g, we have as $n \longrightarrow \infty$:

$$gx_n \longrightarrow F(x,y), \quad gy_n \longrightarrow F(y,x),$$

and

$$gx_n^* \longrightarrow F(x^*,y^*), \quad gy_n^* \longrightarrow F(y^*,x^*).$$

Since

$$\big(F(x,y), F(y,x)\big) = (gx, gy)$$

and

$$\big(F(u,v), F(v,u)\big) = (gu_1, gv_1)$$

are comparable, then $gx \sqsubseteq gu_1$ and $gy \sqsupseteq gv_1$. Similarly, we can show that (gx, gy) and (gu_n, gv_n) are comparable for all $n \geq 1$, that is, $gx \sqsubseteq gu_n$ and $gy \sqsupseteq gv_n$. Now, we can apply the contractive condition:

$$\begin{aligned} d(gu_{n+1}, gx) &= d\big(F(u_n,v_n), F(x,y)\big) \\ &\preceq \big[a_1 d(gu_n, F(u_n,v_n)) + a_2 d(gv_n, F(v_n,u_n))\big] \\ &\quad + \big[a_3 d(gx, F(x,y) + a_4 d(gy, F(y,x))\big] \\ &\quad + \big[a_5 d(gu_n, F(x,y)) + a_6 d(gv_n, F(y,x))\big] \\ &\quad + \big[a_7 d(gx, F(u_n,v_n)) + a_8 d(gy, F(v_n,u_n))\big] \\ &\quad + \big[a_9 d(gu_n, gx) + a_{10} d(gv_n, gy)\big] \\ &= \big[a_1 d(gu_n, gu_{n+1}) + a_2 d(gv_n, gv_{n+1})\big] \\ &\quad + \big[a_3 d(gx, gx) + a_4 d(gy, gy)\big] \\ &\quad + \big[a_5 d(gu_n, gx) + a_6 d(gv_n, gy)\big] \\ &\quad + \big[a_7 d(gx, gu_{n+1}) + a_8 d(gy, gv_{n+1})\big] \\ &\quad + \big[a_9 d(gu_n, gx) + a_{10} d(gv_n, gy)\big] \\ &\preceq \big[sa_1 d(gu_n, gx) + sa_1 d(gx, gu_{n+1}) \\ &\quad + sa_2 d(gv_n, gy) + sa_2 d(gy, gv_{n+1})\big] \\ &\quad + \big[a_5 d(gu_n, gx) + a_6 d(gv_n, gy)\big] \\ &\quad + \big[a_7 d(gx, gu_{n+1}) + a_8 d(gy, gv_{n+1})\big] \\ &\quad + \big[a_9 d(gu_n, gx) + a_{10} d(gv_n, gy)\big]. \end{aligned}$$

Hence

$$\begin{aligned} d(gu_{n+1}, gx) \preceq\ & \big[(sa_1 + a_5 + a_9) d(gu_n, gx) \\ & + (sa_1 + a_7) d(gu_{n+1}, gx)\big] \\ & + \big[(sa_2 + a_6 + a_{10}) d(gv_n, gy) \\ & + (sa_2 + a_8) d(gv_{n+1}, gy)\big]. \end{aligned} \tag{3.9}$$

By similar way, we have

$$\begin{aligned} d(gv_{n+1}, gy) \preceq\ & \big[(sa_1 + a_5 + a_9) d(gv_n, gy) \\ & + (sa_1 + a_7) d(gv_{n+1}, gy)\big] \\ & + \big[(sa_2 + a_6 + a_{10}) d(gu_n, gx) \\ & + (sa_2 + a_8) d(gu_{n+1}, gx)\big]. \end{aligned} \tag{3.10}$$

Put, $\tau_n = d(gu_{n+1}, gx) + d(gv_{n+1}, gy)$. Adding above inequalities, we get

$$\begin{aligned} \tau_n \preceq\ & \big[(sa_1 + a_5 + a_9 + sa_2 + a_6 + a_{10})\tau_{n-1}\big] \\ & + \big[(sa_1 + a_7 + sa_2 + a_8)\tau_n\big]. \end{aligned} \tag{3.11}$$

On the other hand starting by gx, we have:

$$\begin{aligned} & d(gx, gu_{n+1}) = d\big(F(x,y), F(u_n,v_n)\big) \\ & \preceq \big[a_1 d(gx, F(x,y) + a_2 d(gy, F(y,x))\big] \\ & \quad + \big[a_3 d(gu_n, F(u_n,v_n)) + a_4 d(gv_n, F(v_n,u_n))\big] \\ & \quad + \big[a_5 d(gx, F(u_n,v_n)) + a_6 d(gy, F(v_n,u_n))\big] \\ & \quad + \big[a_7 d(gu_n, F(x,y)) + a_8 d(gv_n, F(y,x))\big] \\ & \quad + \big[a_9 d(gx, gu_n) + a_{10} d(gy, gv_n)\big] \\ & = \big[a_1 d(gx, gx) + a_2 d(gy, gy)\big] \\ & \quad + \big[a_3 d(gu_n, gu_{n+1}) + a_4 d(gv_n, gv_{n+1})\big] \\ & \quad + \big[a_5 d(gx, gu_{n+1}) + a_6 d(gy, gv_{n+1})\big] \\ & \quad + \big[a_7 d(gu_n, gx) + a_8 d(gv_n, gy)\big] \\ & \quad + \big[a_9 d(gx, gu_n) + a_{10} d(gy, gv_n)\big] \\ & \preceq \big[sa_3 d(gu_n, gx) + sa_3 d(gx, gu_{n+1}) \\ & \quad + sa_4 d(gv_n, gy) + sa_4 d(gy, gv_{n+1})\big] \\ & \quad + \big[a_5 d(gx, gu_{n+1}) + a_6 d(gy, gv_{n+1})\big] \\ & \quad + \big[a_7 d(gu_n, gx) + a_8 d(gv_n, gy)\big] \\ & \quad + \big[a_9 d(gx, gu_n) + a_{10} d(gy, gv_n)\big] \end{aligned}$$

Hence

$$\begin{aligned} d(gx, gu_{n+1}) \preceq\ & \big[(sa_3 + a_7 + a_9) d(gu_n, gx) \\ & + (sa_3 + a_5) d(gu_{n+1}, gx)\big] + \big[(sa_4 + a_8 + a_{10}) d(gv_n, gy) \\ & + (sa_4 + a_6) d(gv_{n+1}, gy)\big]. \end{aligned} \tag{3.12}$$

By similar way, we have

$$\begin{aligned} d(gy, gv_{n+1}) \preceq & \left[(sa_3 + a_7 + a_9)d(gv_n, gy)\right. \\ & \left. + (sa_3 + a_5)d(gv_{n+1}, gy)\right] \\ & + \left[(sa_4 + a_8 + a_{10})d(gu_n, gx)\right. \\ & \left. + (sa_4 + a_6)d(gu_{n+1}, gx)\right]. \end{aligned} \tag{3.13}$$

Adding above two inequalities, we get

$$\begin{aligned} \tau_n \preceq & \left[(sa_3 + a_7 + a_9 + sa_4 + a_8 + a_{10})\tau_{n-1}\right] \\ & + \left[(sa_3 + a_5 + sa_4 + a_6)\tau_n\right]. \end{aligned} \tag{3.14}$$

Adding inequalities (3.11) and (3.14), we have

$$\begin{aligned} 2\tau_n \preceq & \left[(sa_1 + sa_2 + sa_3 + sa_4) + (a_5 + a_6 + a_7 + a_8)\right. \\ & \left. + 2(a_9 + a_{10})\right]\tau_{n-1} + \left[(sa_1 + sa_2 + sa_3 + sa_4)\right. \\ & \left. + (a_5 + a_6 + a_7 + a_8)\right]\tau_n, \end{aligned}$$

which implies that

$$\tau_n \preceq k\tau_{n-1},$$

where $k = \frac{(sa_1+sa_2+sa_3+sa_4)+(a_5+a_6+a_7+a_8)+2(a_9+a_{10})}{2-\left[(sa_1+sa_2+sa_3+sa_4)+(a_5+a_6+a_7+a_8)\right]} < 1.$

Note that, $\frac{(sa_1+sa_2+sa_3+sa_4)+(a_5+a_6+a_7+a_8)+2(a_9+a_{10})}{2-\left[(sa_1+sa_2+sa_3+sa_4)+(a_5+a_6+a_7+a_8)\right]} < 1$ equivalently $s(a_1 + a_2 + a_3 + a_4) + (a_5 + a_6 + a_7 + a_8 + a_9 + a_{10}) < 1.$

Now, we have

$$\begin{aligned} \tau_n & \preceq k\tau_{n-1} \\ & \preceq k^2\tau_{n-2} \\ & \vdots \\ & \preceq k^n\tau_0 \quad \longrightarrow \theta \quad \text{as} \quad n \longrightarrow \infty. \end{aligned} \tag{3.15}$$

According to Lemma 2.3 (2), and for any $c \in E$ with $c \gg \theta$, there exists $N_0 \in \mathbb{N}$ such that for any $n > N_0$, $k^n \ll c$. Furthermore, from (3.15) and for any $n > N_0$, Lemma 2.3 (3) shows that

$$d(gu_{n+1}, gx) + d(gv_{n+1}, gy) \ll c,$$

which implies that

$$d(gu_{n+1}, gx) \ll c,$$

and

$$d(gv_{n+1}, gy) \ll c.$$

Hence, by Definition 2.2 (1), $gu_n \longrightarrow gx$ and $gv_n \longrightarrow gy$. By the same way, we can prove that $gu_n \longrightarrow gx^*$ and $gv_n \longrightarrow gy^*$. The uniqueness of the limit implies that $gx = gx^*$ and $gy = gy^*$. That is, the unique coupled point of coincidence of F and g is (gx, gy).

Clearly that if (gx, gy) is a coupled point of coincidence of F and g, then (gy, gx) is also a coupled points of coincidence of F and g. Then $gx = gy$ and therefore (gx, gx) is the unique coupled point of coincidence of F and g.

Now, let $u = gx = F(x, y)$. Since F and g are w-compatible, then we have

$$gu = g(gx) = gF(x, y) = F(gx, gy) = F(gx, gx) = F(u, u).$$

Then (gu, gu) is a coupled point of coincidence and also we have (u, u) is a coupled point of coincidence. The uniqueness of the coupled point of coincidence implies that $gu = u$. Therefore $u = gu = F(u, u)$. Hence (u, u) is the unique common coupled fixed point of F and g. This completes the proof. □

From Theorem 3.5, we have the following corollaries.

Corollary 3.6 *In addition to the hypotheses of Corollary 3.2, suppose that for every $(x, y), (x^*, y^*) \in X \times X$ there exists $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable both to $(F(x, y), F(y, x))$ and $(F(x^*, y^*), F(y^*, x^*))$. If F and g are w-compatible, then F and g have a unique common coupled fixed point. Moreover, a common coupled fixed point of F and g is of the form (u, u) for some $u \in X$.*

Corollary 3.7 *In addition to the hypotheses of Corollary 3.3. Suppose that for every $(x, y), (x^*, y^*) \in X \times X$ there exists $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable both to $(F(x, y), F(y, x))$ and $(F(x^*, y^*), F(y^*, x^*))$. Assume that, $k + l < \frac{1}{s}$. If F and g are w-compatible, then F and g have a unique common coupled fixed point. Moreover, a common coupled fixed point of F and g is of the form (u, u) for some $u \in X$.*

Corollary 3.8 *In addition to the hypotheses of Corollary 3.4. Suppose that for every $(x, y), (x^*, y^*) \in X \times X$ there exists $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable both to $(F(x, y), F(y, x))$ and $(F(x^*, y^*), F(y^*, x^*))$. If F and g are w-compatible, then F and g have a unique common coupled fixed point. Moreover, a common coupled fixed point of F and g is of the form (u, u) for some $u \in X$.*

Remark 3.9 Theorems 3.1 and 3.5 extend and generalize Theorems 3.1 and 3.2 of Nashine et al. [3] to cone b-metric spaces, respectively.

Now, we present one example to illustrate our results.

Example 3.10 Let $X = \mathbb{R}$ be ordered by the following relation:

$$x \sqsubseteq y \iff x = y \quad \text{or} \quad \left(x, y \in [0, 1] \quad \text{and} \quad x \le y\right).$$

Let $E = C^1_{\mathbb{R}}[0, 1]$ with $\|u\| = \|u\|_\infty + \|u'\|_\infty, u \in E$ and suppose that, $P = \{u \in E : u(t) \ge 0 \text{on} [0, 1]\}$. It is well known that this cone is solid, but it is not normal. Define a cone b-metric $d : X \times X \to E$ by $d(x, y)(t) = |x - y|^2 e^t$. Then (X, d) is a complete cone b-metric space with the coefficient $s=2$. Let $F: X\times X\to X$ $F(x,y)=\frac{x-y}{60}$ and define $g: X\to X$ by:

$$g(x)=\begin{cases}\frac{1}{2}x, & \text{if } x<0,\\ x, & \text{if } x\in[0,1],\\ \frac{1}{2}x+\frac{1}{3}, & \text{if } x>1.\end{cases}$$

We will check that conditions of theorems 3.1 and 3.5 are fulfilled for all $x,y,u,v\in X$ with ($gu\sqsubseteq gx$ and $gv\sqsupseteq gy$) or ($gx\sqsubseteq gu$ and $gy\sqsupseteq gv$). The following cases are possible.

case 1: $x,y,u,v\in[0,1]$. We have

$$\begin{aligned} d\big(F(x,y),F(u,v)\big)(t) &= \left|\frac{x-y}{60}-\frac{u-v}{60}\right|^2 e^t\\ &\preceq 2\left(\frac{1}{60}|x-u|^2e^t+\frac{1}{60}|y-v|^2e^t\right)\\ &=\frac{1}{30}|x-u|^2e^t+\frac{1}{30}|y-v|^2e^t\\ &=a_9d(gx,gu)(t)+a_{10}d(gy,gv)(t),\end{aligned}$$

where $a_9=\frac{1}{30}=a_{10}$ and $a_i=0, i=1,2,\ldots,8$.

case 2: $x,y\in[0,1]$ and u, v not in [0, 1]. Then gy, gv not in [0, 1] and since they must be comparable, $gy=gv$ and $y=v$. Then we have:

$$\begin{aligned} d\big(F(x,y),F(u,v)\big)(t) &= \left|\frac{x-u}{60}\right|^2 e^t\\ &\preceq a_9d(gx,gu)(t)+a_{10}d(gy,gv)(t),\end{aligned}$$

where $a_9=\frac{1}{30}=a_{10}$ and $a_i=0, i=1,2,\ldots,8$.

case 3: $u,v\in[0,1]$ and x, y not in [0, 1]. This case will be similar to case 2.

case 4: If x, y, u, v not in [0,1] then the only possibility for gx and gu, as well as gy and gv to be comparable is that $x=u$ and $y=v$. In this case conditions of Theorem 3.1 are trivially satisfied.

Note that, $2s(a_9+a_{10})=4(\frac{1}{30}+\frac{1}{30})<2$, $F(X\times X)\subseteq g(X)$ and $g(X)$ is a complete subspace of X. Also, F has the mixed g-monotone property. Hence, the conditions of Theorem 3.1 are satisfied, that is, F and g have a coupled coincidence point (0, 0). Also, F and g are w-compatible at (0, 0) and $a_9+a_{10}<1$. Hence, Theorem 3.5 shows that, (0, 0) is the unique common coupled fixed point of F and g.

Finally, we have the following result (immediate consequence of Theorems 3.1 and 3.5).

Theorem 3.11 *Let $(X,\sqsubseteq)$ be a partially ordered set and (X, d) be a complete cone b-metric space with the coefficient $s\geq 1$ relative to a solid cone P. Let $F: X^2\longrightarrow X$ be a mapping having the mixed monotone property on X and suppose that there exist nonnegative constants $a_i\in[0,1), i=1,2,\ldots,10$ with $\sum_{i=1}^{10}a_i<1$ such that the following contractive condition holds*

$$\begin{aligned} d(F(x,y),F(u,v)) \preceq{} & \big[a_1d(x,F(x,y))+a_2d(y,F(y,x))\big]\\ &+\big[a_3d(u,F(u,v))+a_4d(v,F(v,u))\big]\\ &+\big[a_5d(x,F(u,v))+a_6d(y,F(v,u))\big]\\ &+\big[a_7d(u,F(x,y))+a_8d(v,F(y,x))\big]\\ &+\big[a_9d(x,u)+a_{10}d(y,v)\big],\end{aligned}$$

for all $(x,y),(u,v)\in X^2$ with ($u\sqsubseteq x$ and $v\sqsupseteq y$) or ($x\sqsubseteq u$ and $y\sqsupseteq v$) such that:

(A) $(s+1)(a_1+a_2+a_3+a_4)+s(s+1)(a_5+a_6+a_7+a_8)+2s(a_9+a_{10})<2$

(B) $s(a_1+a_2+a_3+a_4)+(a_5+a_6+a_7+a_8+a_9+a_{10})<1$.

Suppose that X has the following properties:

(i) if a non-decreasing sequence $\{x_n\}$ in X is such that $x_n\longrightarrow x$, then $x_n\sqsubseteq x$ for all $n\in\mathbb{N}$,

(ii) if a non-increasing sequence $\{y_n\}$ in X is such that $y_n\longrightarrow y$, then $y_n\sqsupseteq y$ for all $n\in\mathbb{N}$.

If there exist $x_0,y_0\in X$ such that $x_0\sqsubseteq F(x_0,y_0)$ and $F(y_0,x_0)\sqsubseteq y_0$, then F has a coupled fixed point $(x^*,y^*)\in X^2$. Moreover, the coupled fixed point is unique and of the form (x^*,x^*) for some $x^*\in X$.

The following corollaries can be obtained from Theorem 3.11.

Corollary 3.12 *Let $(X,\sqsubseteq)$ be a partially ordered set and (X, d) be a complete cone b-metric space with the coefficient $s\geq 1$ relative to a solid cone P. Let $F: X^2\longrightarrow X$ be a mapping having the mixed monotone property on X and suppose that there exist nonnegative constants $k,l\in(0,1]$ with $k+l<\frac{1}{s}$ such that the following contractive condition holds*

$$d\big(F(x,y),F(u,v)\big)\preceq kd(x,u)+ld(y,v),$$

for all $(x,y),(u,v)\in X^2$ with ($u\sqsubseteq x$ and $v\sqsupseteq y$) or ($x\sqsubseteq u$ and $y\sqsupseteq v$). Suppose that X has the following properties:

(i) if a non-decreasing sequence $\{x_n\}$ in X is such that $x_n\longrightarrow x$, then $x_n\sqsubseteq x$ for all $n\in\mathbb{N}$,

(ii) if a non-increasing sequence $\{y_n\}$ in X is such that $y_n\longrightarrow y$, then $y_n\sqsupseteq y$ for all $n\in\mathbb{N}$.

If there exist $x_0,y_0\in X$ such that $x_0\sqsubseteq F(x_0,y_0)$ and $F(y_0,x_0)\sqsubseteq y_0$, then F has a coupled fixed point $(x^*,y^*)\in X^2$. Moreover, the coupled fixed point is unique and of the form (x^*,x^*) for some $x^*\in X$.

Corollary 3.13 *Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) be a complete cone b-metric space with the coefficient $s \geq 1$ relative to a solid cone P. Let $F : X^2 \longrightarrow X$ be a mapping having the mixed monotone property on X and suppose that there exist nonnegative constants $k, l \in (0, 1]$ with $k + l < \frac{1}{s}$ such that the following contractive condition holds*

$$d\big(F(x,y), F(u,v)\big) \preceq kd(x, F(x,y)) + ld(u, F(u,v)),$$

for all $(x,y),(u,v) \in X^2$ with ($u \sqsubseteq x$ and $v \sqsupseteq y$) or ($x \sqsubseteq u$ and $y \sqsupseteq v$). Suppose that X has the following properties:

(i) if a non-decreasing sequence $\{x_n\}$ in X is such that $x_n \longrightarrow x$, then $x_n \sqsubseteq x$ for all $n \in \mathbb{N}$,
(ii) if a non-increasing sequence $\{y_n\}$ in X is such that $y_n \longrightarrow y$, then $y_n \sqsupseteq y$ for all $n \in \mathbb{N}$.

If there exist $x_0, y_0 \in X$ such that $x_0 \sqsubseteq F(x_0, y_0)$ and $F(y_0, x_0) \sqsubseteq y_0$, then F has a coupled fixed point $(x^*, y^*) \in X^2$. Moreover, the coupled fixed point is unique and of the form (x^*, x^*) for some $x^* \in X$.

Note that, in above corollary, we ignore condition $k + l < \frac{2}{s+1}$ because $\frac{1}{s} \leq \frac{2}{s+1}$.

Corollary 3.14 *Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) be a complete cone b-metric space with the coefficient $s \geq 1$ relative to a solid cone P. Let $F : X^2 \longrightarrow X$ be a mapping having the mixed monotone property on X and suppose that there exist nonnegative constants $k, l \in (0, 1]$ with $k + l < \frac{2}{s(s+1)}$ such that the following contractive condition holds*

$$d\big(F(x,y), F(u,v)\big) \preceq kd\big(x, F(x,y)\big) + ld\big(u, F(u,v)\big),$$

for all $(x,y),(u,v) \in X^2$ with ($u \sqsubseteq x$ and $v \sqsupseteq y$) or ($x \sqsubseteq u$ and $y \sqsupseteq v$). Suppose that X has the following properties:

(i) if a non-decreasing sequence $\{x_n\}$ in X is such that $x_n \longrightarrow x$, then $x_n \sqsubseteq x$ for all $n \in \mathbb{N}$,
(ii) if a non-increasing sequence $\{y_n\}$ in X is such that $y_n \longrightarrow y$, then $y_n \sqsupseteq y$ for all $n \in \mathbb{N}$.

If there exist $x_0, y_0 \in X$ such that $x_0 \sqsubseteq F(x_0, y_0)$ and $F(y_0, x_0) \sqsubseteq y_0$, then F has a coupled fixed point $(x^*, y^*) \in X^2$. Moreover, the coupled fixed point is unique and of the form (x^*, x^*) for some $x^* \in X$.

Example 3.15 Let $X = \mathbb{R}$ with usual order and let a cone b-metric d defined as in Example 3.10,

$$d(x,y)(t) = |x - y|^2 \cdot \mathrm{e}^t.$$

Then (X, d) is a cone b-metric space with the coefficient $s = 2$. Now, let $F : X \times X \rightarrow X$ as

$$F(x,y) = \frac{x - 2y}{8}.$$

We shall check that this example satisfies all conditions of Corollary 3.12.

Indeed, we have

$$\begin{aligned}
&d\big(F(x,y), F(u,v)\big)(t)\\
&= \left|\frac{x-u}{8} - \frac{2(y-v)}{8}\right|^2 \cdot \mathrm{e}^t\\
&\leq \frac{1}{64}\Big(|x-u|^2 + 4|y-v|^2 + 4|x-u||y-v|\Big) \cdot \mathrm{e}^t\\
&\leq \frac{1}{64}\Big(|x-u|^2 + 4|y-v|^2 + 2\Big(|x-u|^2 + |y-v|^2\Big)\Big) \cdot \mathrm{e}^t\\
&= \frac{3}{64}|x-u|^2 \cdot e^t + \frac{6}{64}|y-v|^2 \cdot \mathrm{e}^t,
\end{aligned}$$

for all $x, y, u, v \in X$ with $x \leq u$ and $y \leq v$ or $x \geq u$ and $y \leq v$ and for all $t \in [0, 1]$. Taking $k = \frac{3}{64}, l = \frac{6}{64}$ we get

$$d(F(x,y), F(u,v)) \leq kd(x,u) + ld(y,v).$$

Since $k + l = \frac{9}{64} < \frac{1}{2} = \frac{1}{s}$, we have that this example of ordered cone b-metric space over (only) solid cone supports Corollary 3.12. Here, $(0, 0)$ is (even) unique coupled fixed point.

Remark 3.16 It is worth to notice that using already some known methods (for example see [24, 25]) contractive condition from Theorem 3.1. implies the following contractive condition in ordered cone b-metric space $(X \times X, D, \sqsubseteq)$:

$$\begin{aligned}
D(T_F Y, T_F V) \leq{} & A_1 D\big(T_g Y, T_F Y\big) + A_2 D\big(T_g V, T_F V\big)\\
&+ A_3 D\big(T_g Y, T_F V\big) + A_4 D\big(T_g V, T_F Y\big) + A_5 D\big(T_g Y, T_g V\big),
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= a_1 + a_2, A_2 = a_3 + a_4, A_3 = a_5 + a_6, A_4\\
&= a_7 + a_8, A_5 = a_9 + a_{10},
\end{aligned}$$

with

$$(s+1)(A_1 + A_2) + s(s+1)(A_3 + A_4) + 2sA_2 < 2,$$

and

$$D((x,y),(u,v)) = d(x,u) + d(y,v).$$

It is not hard to check that $(X \times X, D)$ is a new cone b-metric space with the same coefficient s as (X, d). Also, $(X \times X, D)$ is a ordered cone b-metric space. It is clear that the approach with new ordered cone b-metric space is much shorter, but both ordered cases are new.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no competing interests.

References

1. Ran, A.C.M., Reurings, M.C.B.: A fixed point theorem in partially ordered sets and some applications to matrix equations. Proc. Am. Math. Soc. **132**(5), 1435–1443 (2004)
2. Lakshmikantham, V., Ćirić, L.: Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces. Nonlinear Anal. **70**(12), 4341–4349 (2009)
3. Nashine, H.K., Kadelburg, Z., Radenović, S.: Coupled common fixed point theorems for w*-compatible mappings in ordered cone metric spaces. Appl. Math. Comput. **218**, 5422–5432 (2012)
4. Bhaskar, T.G., Lakshmikantham, V.: Fixed point theorems in partially ordered metric spaces and applications. Nonlinear Anal. **65**(7), 1379–1393 (2006)
5. Abbas, M., Khan, M.A., Radenović, S.: Common coupled fixed point theorems in cone metric spaces for w-compatible mappings. Appl. Math. Comput. **217**, 195–202 (2010)
6. Nashine, H.K., Abbas, M.: Common fixed point of mappings satisfying implicit contractive conditions in TVS-valued ordered cone metric spaces. J. Nonlinear Sci. Appl. **6**, 205–215 (2013)
7. Nashine, H.K., Aydi, H.: Common fixed point theorems for four mappings involving generalized contractions in cone metric spaces. J. Nonlinear Anal. Appl. **2012**, 1–12 (2012)
8. Nashine, H.K., Aydi, H.: Common fixed points for generalized (ψ,φ)-weak contractions in ordered cone metric spaces. Appl. Gen. Topol. **13**(2), 151–166 (2012)
9. Samet, B., Vetro, C.: Coupled fixed point, f-invariant set and fixed point of N-order. Ann. Funct. Anal. **1**(2), 46–56 (2010)
10. Shatanawi, W., Nashine, H.N.: A generalization of Banach's contraction principle for nonlinear contraction in a partial metric space. J. Nonlinear Sci. Appl. **5**, 37–43 (2012)
11. Samet, B., Karapinar, E., Aydi, H., Rajic, C.: Discussion on some coupled fixed point theorems. Fixed Point Theory Appl. **2013**, 50 (2013)
12. Sabetghadam, F., Masiha, H.P., Sanatpour, A.H.: Some coupled fixed point theorems in cone metric space. Fixed Point Theory Appl. **2009**(Article ID 125426) (2009). doi:10.1155/2009/125426
13. Aydi, H., Karapinar, E., Shatanawi, W.: Coupled fixed point results for (ϕ,ψ)-weakly contractive condition in ordered partial metric spaces. Comput. Math. Appl. **62**, 4449–4460 (2011)
14. Aydi, H., Samet, B., Vetro, C.: Coupled fixed point results in cone metric spaces for w-compatible mappings. Fixed Point Theory Appl. **2011**, 27 (2011)
15. Huang, L.G., Zhang, X.: Cone metric spaces and fixed point theorems of contractive mappings. J. Math. Anal. Appl. **332**(2), 1468–1476 (2007)
16. Hussain, N., Shah, M.H.: KKM mappings in cone b-metric spaces. Comput. Math. Appl. **62**(4), 1677–1684 (2011)
17. Radenović, S., Kadelburg, Z.: Quasi-contractions on symmetric and cone symmetric spaces. Banach J. Math. Anal. **5**(1), 38–50 (2011)
18. Cvetković, A.S., Stanić, M.P., Dimitrijević, S., Simić, S.: Common fixed point theorems for four mappings on cone metric type space. Fixed Point Theory Appl. **2011**, Article ID 589725 (2011)
19. Stanić, M.P., Cvetković, A.S., Simić, S., Dimitrijević, S.: Common fixed point under contractive condition of Cirics type on cone metric type spaces. Fixed Point Theory Appl. **2012**, 35 (2012)
20. Shah, M.H., Simić, S., Hussain, N., Sretenović, A., Radenović, S.: Common fixed points for occasionally weakly compatible pairs on cone metric type spaces. J. Comput. Anal. Appl. **14**(2), 290–297 (2012)
21. Huang, H., Xu, S.: Fixed point theorems of contractive mappings in cone b-metric spaces and applications. Fixed Point Theory Appl. **2012**, 220 (2012)
22. Shi, L., Xu, S.: Common fixed point theorems for two weakly compatible self-mappings in cone b-metric spaces. Fixed Point Theory Appl. **2013**, 120 (2013)
23. Fadail, Z.M., Ahmad, A.G.B.: Coupled coincidence point and common coupled fixed point results in cone b-metric spaces. Fixed Point Theory Appl. **2013**, 177 (2013)
24. Du, W., Karapnar, E.: A note on cone b-metric and its related results: generalizations or equivalence? Fixed Point Theory Appl. **2013**, 210 (2013)
25. Popović, B., Radenović, S., Shukla, S.: Fixed point results to tvs-cone b-metric spaces. Gulf J. Math. **1**, 51–64 (2013)
26. Azam, A., Mehmood, N., Ahmad, J., Radenović, S.: Multivalued fixed point theorems in cone b-metric spaces. J. Inequal. Appl. **2013**, 582 (2013)
27. Rad, G.S., Aydi, H., Kumam, P., Rahimi, H.: Common tripled fixed point results in cone metric type spaces. Rend. Circ. Mat. Palermo **63**, 287–300 (2014)
28. Kumam, P., Rahimi, H., Soleimani Rad, G.: The existence of fixed and periodic point theorems in cone metric type spaces. J. Nonlinear Sci. Appl. **7**, 255–263 (2014)
29. Rahimi, H., Soleimani Rad, G., Kumam, P.: Coupled common fixed point theorems under weak con tractions in cone metric type spaces. Thai. J. Math. **12**(1), 1–14 (2014)
30. Agarwal, R.P., El-Gebeily, M.A., O'Regan, D.: Generalized contractions in partially ordered metric spaces. Appl. Anal. **87**, 1–8 (2008)
31. O'Regan, D., Petrusel, A.: Fixed point theorems for generalized contractions in ordered metric spaces. J. Math. Anal. Appl. **341**(2), 1241–1252 (2008)
32. Altun, I., Damjanović, B., Djorić, D.: Fixed point and common fixed point theorems on ordered cone metric spaces. Appl. Math. Lett. **23**, 310–316 (2009)
33. Kadelburg, Z., Pavlović, M., Radenović, S.: Common fixed point theorems for ordered contractions and quasicontractions in ordered cone metric spaces. Comput. Math. Appl. **59**, 3148–3159 (2010)
34. Aydi, H., Nashine, H.K., Samet, B., Yazidi, H.: Coincidence and common fixed point results in partially ordered cone metric spaces and applications to integral equations. Nonlinear Anal. **74**, 6814–6825 (2011)
35. Radenović, S.: Some coupled coincidence points results of monotone mappings in partially ordered metric spaces. Int. J. Anal. Appl. **5**(2), 174–184 (2014)
36. Radenović, S.: Bhaskar-Lakshmikantham type-results for monotone mappings in partially ordered metric spaces. Int. J. Nonlinear Anal. Appl. **5**(2), 37–49 (2014)
37. Radenović, S.: Remarks on some coupled fixed point results in partial metric spaces. Nonlinear Funct. Anal. Appl. **18**(1), 39–50 (2013)
38. Fadail, Z.M., Ahmad, A.G.B.: Fixed point theorem of N-order in cone metric spaces. Far East J. Math. Sci. **79**(1), 49–76 (2013)

39. Fadail, Z.M., Ahmad, A.G.B.: Common fixed point theorem of N-order for W_N-compatible mappings in cone metric spaces. Far East J. Math. Sci. **96**(2), 133–166 (2015)
40. Jungck, G., Radenović, S., Radojević, S., Rakočević, V.: Common fixed point theorems for weakly compatible pairs on cone metric spaces. Fixed Point Theory Appl. **2009**, 1–13 (2009)